MATH 590: QUIZ 12 SOLUTIONS

Name:

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P. Each part is worth 2 points.

(i) Find $p_A(x)$ and the single eigenvalue λ.

Solution. $p_A(x) = \begin{vmatrix} x-1 & -1 & 0 \\ -1 & x-1 & -1 \\ 0 & 1 & x-1 \end{vmatrix} = (x-1)^3$.

(ii) Calculate E_λ.

Solution. $E_1 =$ null space of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \overset{\text{EROs}}{\to} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for E_1.

(iii) Calculate $(A - \lambda I)^2$.

Solution. $(A-I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. The null space of $(A-I)^2$ is the null space of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(iv) Find v_3 not in the null space of $(A-\lambda I)^2$. Take $v_2 := (A-\lambda I)v_3$ and $v_1 := (A-\lambda I)v_2$.

Solution. We can take

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Thus $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(v) Letting P be the matrix whose v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & \lambda & \lambda \end{pmatrix}$.

Solution.

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$